INTEGRAL PRODUCTS AND PROBABILITY.

By P. J. Daniell.

1. *Introduction.*—In many problems arising in statistical biology and statistical economics time enters as an indispensable factor. It is the chief aim of this paper to provide a form of analysis suitable for such problems, and this will be a theory of probability using time as an auxiliary variable. The term "Dynamic Probability" may be used to avoid confusion with Gibbs' Statistical Mechanics, which is a theory of interactions between fast-moving molecules or between vibratory systems, whereas our theory is that of time-variations in position of groups moving more or less erratically. That is to say, the basis is change of position rather than of velocity, and the tendency to disperse is regarded as inherent, not as the effect of "collisions."

For the sake of simplicity we consider only motion in a single dimension. The variable x may be a measure of actual geometrical position, or it may measure some factor of the relative environment, for example, temperature, intensity of light, or it may refer to a quantity of goods.

The first step in the analysis is a search for some standard formula on which may be built a more complex and general theory. It is found that, if certain natural assumptions are made, a functional equation is satisfied, which is expressed in terms of a Stieltjes integral product. This form of product may be compared with Volterra's integral composition, and on it a similar algebra may be built. It is shown that there exists an idem-factor function for this algebra whereas such a *function* does not exist for the Volterra composition.* The Stieltjes integral product itself forms a secondary nucleus for our paper; its investigation will be found in paragraph 3.

2. *Fundamental Equation.*—Denote an increase (possibly negative) in x by y and an increase in time t by u. The motion in the interval u will depend partly on the previous history H of the member considered. Denote by

$$P(y_1, u_1, H_0)$$

* V. Volterra, "Leçons sur les fonctions de lignes." Paris (1913). G. C. Evans, Cambridge Colloquium, New York (1916), p. 117. By introducing an auxiliary algebraic symbol j, Evans modifies the Volterra algebra so as to contain an idem-factor but this factor, $1 + j0$, is independent of the variables. For a resumé of the Stieltjes integral and references the reader is referred to T. H. Hildebrandt, *Bulletin of the American Mathematical Society* (1917), 24, p. 117 and p. 177.

the probability that, in an interval u_1, the increase in x does not exceed y_1 (*i.e.*, $y \leqq y_1$) for a certain history H_0 up to time t_0. Similarly we may denote by

$$P(y_2, u_2, H_1)$$

the probability that, in a succeeding interval u_2, the increase in x does not exceed y_2. H_1 will include both H_0 and the motion during the interval u_1.

This is the critical point at which the dynamics of material and living objects diverge. Theoretical mechanics is expressed in terms of velocities and accelerations which are instantaneous time-derivatives. In such a theory, however small we may choose the interval u_1, the second probability $P(y, u_2, H_1)$ depends essentially on the motion during u_1 rather than on H_0. If only u_1 is chosen sufficiently small the contrary is true of living objects. It is to be understood that we are not concerned with the ultimate hypotheses of materialism and vitalism; we regard each object as a whole and do not pretend to delve into the dynamics of its finest constituents which may or may not be of the material type. Our point will be made clear by an illustration. Consider a salmon ascending a stream and suddenly brought to rest by a glass plate, inserted in the stream for a brief instant. After the plate is removed the salmon will resume its motion almost as if there had been no interruption. It will be influenced more by previous history or habit than by the interruption, provided the latter is short. Compare this case with that of a cylinder rolling along a horizontal plane. Even after the very shortest interruption the cylinder will remain at rest. It is more influenced by the immediately preceding history. An example which is an apparent exception is even more illuminating. In a speculative stock market the movement of a particular stock depends partly on the preceding movement. But even here the effect is influenced by the events of some hours or minutes rather than by those of the last hundredth of a second, let us say.

We shall assume that if the interval u_1 is sufficiently small, the probability function P depends on H_0 only, so that

$$P(y_2, u_2, H_1) = P(y_2, u_2, H_0).$$

If we confine our attention to a homogeneous group of members having practically the same history H_0, up to the time t_0, then we may omit the symbol H_0 and define $P(y, u)$ as the probability that the increase in x does not exceed y in a short interval of time u not far removed from a fixed point of time t_0.

This is the fundamental assumption on which our analysis is based; and it is this which sharply distinguishes dynamic probability from the classical theories of mechanics, statistical or otherwise. After an investiga-

tion of $P(y, u)$ in which u is assumed to be small it is possible to build up a more general theory suitable for longer intervals, somewhat as organic chemistry is built on the atomic laws of simpler reactions.

By its definition $P(y, u)$ is a limited non-decreasing function of y approaching 0 as y approaches $-\infty$, and 1 as y approaches $+\infty$. It is also "continuous on the right," that is

$$P(y, u) = \lim_{e \doteq 0,\, e>0} P(y + e, u)$$

$$= P(y + 0, u).$$

On the other hand,

$$P(y - 0, u) = \lim_{e \doteq 0,\, e>0} P(y - e, u)$$

is the probability that the increase in x is *strictly less* than y, and it is not necessarily equal to $P(y, u)$.

The increase y in the interval $u_1 + u_2$ is the sum of an increase z during u_1 and an increase $y - z$ during u_2. Keeping y fixed the function $P(y - z, u_2)$ is a limited non-increasing function of z continuous on the left with the values 1 at $-\infty$, 0 at $+\infty$, and is continuous in each of a countable set of intervals which are open below and closed above.

Let cd be one of these intervals; then the probability that, in the time u_1, the increase z in x satisfies $c < z \leq d$ will be

$$\Delta P(z, u_1) \equiv P(d, u_1) - P(c, u_1).$$

Given that $c < z \leq d$, the probability that the increase in the time $u_1 + u_2$ does not exceed y will lie between

$$P(y - d, u_2),\ P(y - c, u_2).$$

Then the combined probability will lie between

$$P(y - d, u_2)\Delta P(z, u_1),\ P(y - c, u_2)\Delta P(z, u_1).$$

The law of composite probability is applicable here because we assumed the motion during u_2 independent of that in u_1.

Divide cd into smaller sub-intervals and proceed to the limit as their maximum length approaches 0. The combined probability that $c < z \leq d$ and that during $u_1 + u_2$ the increase does not exceed y will be

$$\int_c^d P(y - z, u_2) d_z P(z, u_1).$$

Add up the corresponding expressions for all the intervals cd, and then the fundamental equation is obtained, namely

2(1) $$P(y, u_1 + u_2) = \int_{-\infty}^{+\infty} P(y - z, u_2) d_z P(z, u_1).$$

Without changes in the processes of reasoning we may regard u_2 as the earlier, u_1 the later interval of time and so obtain

$$2(2) \qquad P(y,\, u_1 + u_2) = \int_{-\infty}^{+\infty} P(y - z,\, u_1) d_z P(z,\, u_2).$$

3. *Stieltjes Integral Products.*—The function $\alpha(x)$ will be said to be a regular function of limited variation in x if it is of limited variation $(-\infty$ to $+\infty)$, if $\alpha(-\infty) = 0$ and if, for all values of x,

$$\alpha(x) = \tfrac{1}{2}[\alpha(x + 0) + \alpha(x - 0)].$$

We shall require an important lemma.

3(1). If $\alpha(s, t)$ is a regular function of limited variation in s, and measurable in the sense of Borel in t, and if the variation be limited uniformly with respect to t, and if $g(t)$ is of limited variation and $f(s)$ is bounded and measurable,* then

$$\int_{-\infty}^{+\infty} f(s) d_s \int_{-\infty}^{+\infty} \alpha(s, t) gd(t) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right] dg(t).$$

Let $f(s) = 1$, $(c \leq s \leq d) = 0$ otherwise.

$$\int_{-\infty}^{+\infty} f(s) d_s \int_{-\infty}^{+\infty} \alpha(s, t) dg(t) = \int_{-\infty}^{+\infty} \alpha(d + 0, t) dg(t) - \int_{-\infty}^{+\infty} \alpha(c - 0, t) dg(t)$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right] dg(t).$$

If $f(s)$ is a step-function (constant over each of a finite set of sub-intervals) it is a linear combination of functions of the above type and therefore the equality will be satisfied. But any measurable function $f(s)$ can be obtained from such step-functions by a finite succession of linear combinations and limiting processes using in all cases bounded sets of functions since $f(s)$ is bounded. Because $\alpha(s, t)$ is of uniformly limited variation a number S exists such that the total variation of $\alpha(s, t)$ does not exceed S. If G is the variation of $g(t)$ then the variation of $\int_{-\infty}^{+\infty} \alpha(s, t) dg(t)$ does not exceed SG. Also if $|f(s)| \leq M$,

$$\left| \int_{-\infty}^{+\infty} f(s) d_s \alpha(s, t) \right| \leq MS.$$

Therefore in each of the required limiting processes the several integrands do not exceed summable functions in absolute value, and at each step the equality is preserved.†

* In this paper the attribute 'measurable' will be taken in the sense of Borel.

† P. J. Daniell, *Annals of Mathematics*, Vol. 19 (1918), p. 290. The theorem used here is used frequently in the present paper.

3(2). Let $\alpha_1(x, y)$, $\alpha_2(x, y)$ be regular functions of limited variation in x, and measurable in y, and let their total variations be limited uniformly with respect to y; then we define their $S - V$ composition (Stieltjes-Volterra integral product) as

$$\alpha_1 \cdot \alpha_2(x, y) \equiv \int_{-\infty}^{+\infty} \alpha_1(x, s) d_s \alpha_2(s, y). \quad \textit{Def.}$$

Evidently this "product" satisfies the distributive laws

$$(\alpha_1 + \alpha_2) \cdot \alpha_3 = \alpha_1 \cdot \alpha_3 + \alpha_2 \cdot \alpha_3,$$

$$\alpha_1 \cdot (\alpha_2 + \alpha_3) = \alpha_1 \cdot \alpha_2 + \alpha_1 \cdot \alpha_3.$$

In theorem 3(1) replace $f(s)$ by $\alpha_1(x, s)$, $\alpha(s, t)$ by $\alpha_2(s, t)$ and $g(t)$ by $\alpha_3(t, y)$. The theorem becomes

$$\int_{-\infty}^{\pm\infty} \alpha_1(x, s) d_s \left[\int_{-\infty}^{+\infty} \alpha_2(s, t) d_t \alpha_3(t, y) \right]$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \alpha_1(x, s) d_s \alpha_2(s, t) \right] d_t \alpha_3(t, y),$$

which is, symbolically, the associative law

$$\alpha_1 \cdot (\alpha_2 \cdot \alpha_3) = (\alpha_1 \cdot \alpha_2) \cdot \alpha_3.$$

Consider the function $\eta(x, y) = 0, \quad x < y,$

$$= \tfrac{1}{2}, \quad x = y,$$

$$= 1, \quad x > y.$$

This satisfies the conditions imposed on α_1, α_2, and

$$\alpha \cdot \eta(x, y) = \int_{-\infty}^{+\infty} \alpha(x, s) d_s \eta(s, y) = \alpha(x, y),$$

$$\begin{aligned} \eta \cdot \alpha(x, y) &= \int_{-\infty}^{+\infty} \eta(x, s) d_s \alpha(s, y) \\ &= \alpha(x - 0, y) - \alpha(-\infty, y) + \tfrac{1}{2}[\alpha(x + 0, y) - \alpha(x - 0, y)] \\ &= \alpha(x, y), \end{aligned}$$

since α is a regular function of limited variation in x. We see that η plays the part of an idem-factor in either order,

$$\alpha \cdot \eta = \eta \cdot \alpha = \alpha.$$

Now let η' be another idem-factor satisfying the conditions in 3(2) such that for every α, $\eta' \cdot \alpha = \alpha$. Then by what has been proved $\eta' \cdot \eta = \eta'$ and by

hypothesis $\eta' \cdot \eta = \eta$ so that $\eta' = \eta$. Similarly it can be shown that there is no other idem-fector η'' satisfying the conditions in 3(2) and such that $\alpha \cdot \eta'' = \alpha$ for every α.

Thus η is the only idem-factor in the class of functions α considered.

If it happens that $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_1$ we may say that the functions are S-permutable to distinguish the property from that of being permutable according to Volterra.

If α_1, α_2 are differentiable with respect to their first variables, for example

$$\alpha_1(s, t) = \int_{-\infty}^{s} f_1(s, t) ds, \qquad \alpha_2(s, t) = \int_{-\infty}^{s} f_2(s, t) ds,$$

then

$$\alpha_1 \cdot \alpha_2(s, t) = \int_{-\infty}^{s} \overset{*}{f_1} \overset{*}{f_2}(s, t) ds,$$

in the Volterra notation, and this shows the relation between the two types of combination.

A development of this algebra particularly in connection with integral equations would be interesting but we shall now confine our attention to functions of the difference $x - y$,

$$\alpha(x, y) = \alpha(x - y).$$

Then

$$\alpha_1 \cdot \alpha_2 = \int_{-\infty}^{+\infty} \alpha_1(x - s) d_s \alpha_2(s - y) = \int_{-\infty}^{+\infty} \alpha_1(x - y - z) d_z \alpha_2(z),$$

so that $\alpha_1 \cdot \alpha_2$ is also a function of the difference $x - y$.

3(3). To use symbols more in keeping with our special problem, if $\alpha_1(y)$, $\alpha_2(y)$ are regular functions of limited variation ($-\infty$ to $+\infty$), then we define their $S - V$ product as

$$\alpha_1 \cdot \alpha_2(y) \equiv \int_{-\infty}^{+\infty} \alpha_1(y - z) d\alpha_2(z) \quad \textit{Def.}$$

The corresponding idem-factor will be

$$\begin{aligned} \eta(y) &= 0, \quad y < 0, \\ &= \tfrac{1}{2}, \quad y = 0, \\ &= 1, \quad y > 0. \end{aligned}$$

3(4). THEOREM. *The product as defined in* 3(3) *is S-permutable. For since the functions are regular, by a theorem on integration by parts,**

* P. J. Daniell, *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 362.

$$\alpha_1\cdot\alpha_2(y) = \int_{-\infty}^{+\infty} \alpha_1(y-z)d\alpha_2(z)$$

$$= [\alpha_1(y-z)\alpha_2(z)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \alpha_2(z)d_z\alpha_1(y-z).$$

But $\alpha_1(y-\infty)$, $\alpha_2(-\infty)$ are each 0 while $\alpha_2(+\infty)$, $\alpha_1(y+\infty)$ are finite. Again if $y - z = v$,

$$\Delta_z\alpha_1(y-z) = \alpha_1(y-z_2) - \alpha_1(y-z_1), \qquad z_1 < z_2,$$

$$= \alpha_1(v_2) - \alpha_1(v_1), \qquad v_1 > v_2,$$

$$= -\Delta_v\alpha_1(v).$$

Combining these results,

$$\alpha_1\cdot\alpha_2(y) = \int_{-\infty}^{+\infty} \alpha_2(y-v)d\alpha_1(v) = \alpha_2\cdot\alpha_1(y).$$

This proves the theorem.

In the fundamental equation 2(1) $P(y, u)$ is not regular, but we can replace it by $\beta_u(y) = \frac{1}{2}[P(y+0, u) + P(y-0, u)]$.

Equation 2(1) can now be written

$$P(y, u_1+u_2) = \int_{-\infty}^{+\infty} P(y-z, u_2)d\beta_{u_1}(z),$$

or what is the same equation,

$$P(y+0, u_1+u_2) = \int_{-\infty}^{+\infty} P(y+0-z, u_2)d\beta_{u_1}(z).$$

Again

$$P(y-0, u_1+u_2) = \lim_{e\doteq 0,\ e>0} \int_{-\infty}^{+\infty} P(y-e-z, u_2)d\beta_{u_1}(z)$$

$$= \int_{-\infty}^{+\infty} P(y-0-z, u_2)d\beta_{u_1}(z).$$

This substitution of the integral of the limit for the limit of the integral is legitimate because the integrands are bounded in their set. Adding and dividing by 2, we obtain

$$3(5). \qquad \beta_{u_1+u_2}(y) = \int_{-\infty}^{+\infty} \beta_{u_2}(y-z)d\beta_{u_1}(z) = \beta_{u_2}\cdot\beta_{u_1}(y).$$

4. *Characteristic Function.* We now introduce Poincaré's characteristic function* which is defined as

$$\int_{-\infty}^{+\infty} e^{pz}d\beta(z) \equiv \varphi(p, \beta) \quad \textit{Def.}$$

Before using this function it is necessary to prove some important theorems.

* H. Poincaré, Calcul des Probabilités, Paris (1912), p. 206.

4(1). If $\beta(y)$ is a regular function of limited variation,

$$\int_0^\infty \frac{dk}{\pi k}\int_{-\infty}^{+\infty}[\sin k - \sin(zk - yk)]d\beta(z)$$

exists and is equal to $\beta(y)$.

Consider the integral

$$\int_{k_1}^{k_2} \frac{dk}{\pi k}\int_{-\infty}^{+\infty}[\sin k - \sin(zk - yk)d\beta(z), \qquad k_2 > k_1 > 0.$$

The function $[\sin k - \sin(zk - yk)]/\pi k$ is measurable and bounded ($k_1 \leq k \leq k_2$, $-\infty < z < +\infty$) so that the order of integration may be changed. If we introduce the classical symbol

$$Si(x) = \int_0^x \frac{\sin x}{x}dx$$

the integral can be expressed in the form

$$\int_{-\infty}^{+\infty}[Si(k) - Si(zk - yk)]_{k_1}^{k_2}d\beta(z)/\pi.$$

Since for real values of x, $Si(x)$ is bounded, the integrand is limited and measurable, and the limit of the integral as $k_1 \doteq 0$, $k_2 \doteq \infty$ will exist and will equal the integral of the limit. (We are using again the theorem referred to in 3(1).) But when $k \doteq 0$, $Si(k) \doteq 0$, $Si(zk - yk) \doteq 0$, and when $k \doteq \infty$, $Si(k) \doteq \pi/2$,

$$\begin{aligned} Si(zk - yk) &\doteq \pi/2, & z &> y, \\ &\doteq 0, & z &= y, \\ &\doteq -\pi/2, & z &< y, \end{aligned}$$

so that the limit of the integrand is the idem-factor function $\eta(y - z)$. Therefore the double integral of the theorem exists and is equal to

$$\int_{-\infty}^{+\infty}\eta(y - z)d\beta(z) = \eta\cdot\beta(y) = \beta(y).$$

Corollary.—If $\beta(y)$ is a regular function of limited variation and if $\varphi(p, \beta) = 0$ for all pure imaginary values of $p = ik$, then $\beta(y) = 0$ identically. For then

$$\int_{-\infty}^{+\infty}\sin(zk - yk)d\beta(z) = 0, \qquad k \text{ real},$$

$$\beta(y) = \beta(\infty)\int_0^\infty \frac{dk}{\pi k}\sin k = \tfrac{1}{2}\beta(\infty).$$

Then $\beta(y)$ is constant and since $\beta(-\infty) = 0$, $\beta(y) = 0$ throughout.

4(2). If

$$\int_{-\infty}^{+\infty} e^{hz} d\beta(z), \qquad -H < h < +K,$$

exists, where H, K are *positive*, then so also does

$$\int_{-\infty}^{+\infty} z^n e^{hz} d\beta(z), \qquad -H < h < +K, \qquad n = 1, 2, \cdots.$$

For if $-H < h < K$ we can find a number $\eta > 0$ such that $-H < h - \eta < h + \eta < K$. Given η and n we can find $z_0 > 1$ such that $z^n < e^{\eta z}$, $z \geqq z_0$.

Then $z^n e^{hz} (z \geqq 0)$ is measurable and less than the function $z_0^n e^{(h+\eta)z}$ which is summable β from $-\infty$ to $+\infty$.

Similarly $z^n e^{hz} (z \leqq 0)$ is measurable and its modulus is less than $z_0^n e^{(h-\eta)z}$ which is summable β from $-\infty$ to $+\infty$. Therefore $z^n e^{hz}$ is summable β from $-\infty$ to $+\infty$.

4(3). If

$$\int_{-\infty}^{+\infty} e^{hz} d\beta(z), \qquad -H < h < K,$$

exists then

$$\int_{-\infty}^{+\infty} e^{pz} d\beta(z) = \varphi(p, \beta)$$

is holomorphic in the strip $-H <$ real part of $p < K$. This strip will be called the HK strip.

If $p = h + ik$ lies in this strip, $-H < h < K$ and since $|\cos kz|$, $|\sin kz|$ are not greater than 1, $|e^{hz}| = e^{hz}$, it follows that

$$\varphi(p, \beta) = \int_{-\infty}^{+\infty} e^{hz} \cos kz \, d\beta(z) + i \int_{-\infty}^{+\infty} e^{hz} \sin kz \, d\beta(z)$$

exists.

$$[\varphi(p + \Delta p) - \varphi(p)]/\Delta p = \int_{-\infty}^{+\infty} z e^{pz} \psi(z\Delta p) d\beta(z)$$

in which $\psi(t) = (e^t - 1)/t$.

Integrating along a radius $\int_0^t e^t dt = e^t - 1$, so that $|e^t - 1| \leqq |t| e^{|t|}$, and $|\psi(t)| < e^{|t|}$.

Since $-H < h < K$ we can find a number $\eta > 0$ such that $-H < h - \eta < h + \eta < K$ and if $|\Delta p| \leqq \eta$, the real and imaginary parts of $ze^{pz}\psi(z\Delta p)$ will not be greater in modular values than the function which has for each z the greater of the values of $|z| e^{(p \pm \eta)z}$, which by 4(2) is known to be summable.

Furthermore $\psi(t)$ is continuous at $t = 0$ and its limit there is 1. Therefore by the theorem used in proving 3(1) there exists a unique limit of

$\Delta\varphi/\Delta p$ as Δp approaches 0, that is to say, $\varphi(p)$ is holomorphic in the HK strip and

$$\frac{d\varphi(p)}{dp} = \int_{-\infty}^{+\infty} ze^{pz} d\beta(z).$$

4(4). If $\varphi(h, \beta_1)$, $\varphi(h, \beta_2)$ exist, $-H < h < K$ and if $\beta_1 \cdot \beta_2(y) = \gamma(y)$ then

$$\varphi(p, \gamma) = \varphi(p, \beta_1)\varphi(p, \beta_2)$$

provided p is in the HK strip and conversely.

$$\varphi(p, \gamma) = \int_{-\infty}^{+\infty} e^{py} d_y \int_{-\infty}^{+\infty} \beta_1(y - z) d\beta_2(z)$$

$$= \lim_{Y=\infty} \int_{-Y}^{+Y} e^{py} d_y \int_{-\infty}^{+\infty} \beta_1(y - z) d\beta_2(z)$$

$$= \lim_{Y=\infty} \int_{-\infty}^{+\infty} \left[\int_{-Y}^{+Y} e^{py} d_y \beta_1(y - z) \right] d\beta_2(z)$$

by theorem 3(1), and this

$$= \lim_{Y=\infty} \int_{-\infty}^{+\infty} e^{pz} \left[\int_{-Y+z}^{Y-z} e^{pv} d\beta_1(v) \right] d\beta_2(z).$$

The expression in square brackets is not greater in modular value than $\varphi(h, \omega_1)$ where h is the real part of p, $\omega_1(v)$ is the modular variation function corresponding to $\beta_1(v)$. The existence of $\varphi(h, \beta_1)$ implies that e^{hv} is summable with respect to β_1, and therefore with respect to the variation function ω_1. Therefore using again the theorem with respect to limits under the integral sign

$$\varphi(p, \gamma) = \int_{-\infty}^{+\infty} e^{pz} \varphi(p, \beta_1) d\beta_2(z)$$

$$= \varphi(p, \beta_1)\varphi(p, \beta_2).$$

To prove the converse let $\delta(y) = \beta_1 \cdot \beta_2(y)$, then by the theorem itself

$$\begin{aligned} \varphi(p, \gamma - \delta) &= \varphi(p, \gamma) - \varphi(p, \delta) \\ &= \varphi(p, \gamma) - \varphi(p, \beta_1)\varphi(p, \beta_2) \\ &= 0 \text{ by hypothesis.} \end{aligned}$$

Therefore by 4(1). Cor. $\gamma - \delta = 0$ identically, and $\gamma(y) = \beta_1 \cdot \beta_2(y)$.

4(5). If $\beta_u(y)$ is a regular function of limited variation in y for $u \geqq 0$, if $\beta(y)$ is not identically 0 and if $\varphi(p, \beta_u)$ exists in the HK strip, then if $\varphi(p, \beta_u) = e^{uP}$, where P is holomorphic in the HK strip, and where u is rational and non-negative,

$$\beta_{u_1}\cdot\beta_{u_2}(y) = \beta_{u_1+u_2}(y)$$

and conversely.

The direct theorem is a simple consequence of 4(4) since

$$\varphi(p, \beta_{u_1+u_2}) = e^{(u_1+u_2)P} = \varphi(p, \beta_{u_1})\varphi(p, \beta_{u_2}).$$

To prove the converse let $p = \log_e \varphi(p, \beta_1)$, i.e., where $u = 1$, then by 4(4) since $\beta_{u_1}\cdot\beta_{u_2}(y) = \beta_{u_1+u_2}(y)$,

$$\log \varphi(p, \beta_{u_1}) + \log \varphi(p, \beta_{u_2}) = \log \varphi(p, \beta_{u_1+u_2}),$$

from which, if u is rational and non-negative, we obtain

$$\log \varphi(p, \beta_u) = u \log \varphi(p, \beta_1) = uP, \qquad \varphi(p, \beta_u) = e^{uP}.$$

To complete the theorem it remains to be proved that P is holomorphic in the HK strip. Since $\varphi(p, \beta_1)$ is holomorphic P will be holomorphic except where $\varphi(p, \beta_1) = 0$. Let us suppose that at p_0, $\varphi(p_0, \beta_1) = 0$ then for all non-negative rational u, $\varphi(p_0, \beta_u) = 0$. But if $u = nv$, $\varphi(p, \beta_u) = [\varphi(p, \beta_v)]^n$ when n is a positive integer, and consequently $\delta^r\varphi(b, \beta_u)]_{p=p_0}$, $r < n$, $= 0$, where δ stands for d/dp.

But n can be any positive integer and r any integer less than n so that not only φ but all its derivatives must vanish at p_0. By Taylor series and continuation it follows that $\varphi(p, \beta_u) = 0$ identically over all the HK strip (boundaries excluded) and in particular that it vanishes identically along the imaginary axis. But by 4(1). Cor. this implies that $\beta_u(y) = 0$ identically which is a case excluded by hypothesis. This completes the proof of the theorem.

This theorem enables us to pass from the functional equation

3(5) $$\beta_{u_1}\cdot\beta_{u_2} = \beta_{u_1+u_2}$$

to the algebraic relation $\varphi(p, \beta_u) = e^{uP}$ and back. The formal solution of 3(5) can now be written in the form

4(6) $$\beta_u(y) = \int_0^\infty \frac{dk}{\pi k}[b^u \sin k - e^{uR(k)} \sin \{kI(k) - ky\}]$$

where $R(k) + iI(k) = P(ik) = \log \varphi(ik, \beta_1)$, $b = e^{P(0)} = \varphi(0, \beta_1) = \beta_1(\infty)$ $b^u = e^{uP(0)} = \varphi(0, \beta_u) = \beta_u(\infty)$.

5. *Particular Solutions.*—In our search for the more elementary solutions we shall carry through some processes in a purely formal manner without rigorous justification, but we can test the results obtained. If it is found that $\varphi(p, \beta_u) = e^{uP}$ where P is holomorphic in an HK strip then by 4(5) $\beta_u(y)$ will be a solution of the functional equation 3(5).

In some cases, particularly if $\int_0^\infty e^{uR(k)}dk$ is convergent, we may differentiate both sides of 4(6) and obtain the simplified formula

$$\beta_u(y) = \int_{-\infty}^{y} f_u(y)dy$$

where

$$f_u(y) = \frac{1}{\pi}\int_0^\infty e^{uR(k)} \cos [kI(k) - ky]dk.$$

Let us suppose that $f_u(y)$ satisfies some differential equation which may be written in the form $\Sigma_n\, A_n(D)(y^n f) = 0$, where $D \equiv d/dy$, $A_n(D)$ is a polynomial in the operator D whose coefficients depend on u. Now under certain conditions

$$\int_{-\infty}^{+\infty} e^{pz}z^n f_u(z)dz = \delta^n \int_{-\infty}^{+\infty} e^{pz} f_u(z)dz, \qquad \delta \equiv d/dp,$$

$$= \delta^n e^{uP}.$$

$$\int_{-\infty}^{+\infty} e^{pz}D[Z(z)]dz = [e^{pz}Z]_{-\infty}^{+\infty} - p\int_{-\infty}^{+\infty} e^{pz}Zdz$$

$$= -\, p\int_{-\infty}^{+\infty} e^{pz}Zdz.$$

$$\int_{-\infty}^{+\infty} e^{pz}A_n(D)[z^n f_u(z)]dz = A_n(-\, p)\delta^n e^{uP}.$$

Then P must satisfy the differential equation

$$\Sigma_n A_n(-\, p)\delta^n e^{uP} = 0.$$

This equation is a differential equation for P which is a function of p only and therefore the ratios of the coefficients of any two distinct types of products of P with its derivatives must be independent of u. It follows that the equation can contain only two terms, those for $n = 0$ and $n = 1$. Consider for example the case $n = 2$. Then

$$A_2(-\, p)\delta^2(e^{uP}) = A_2(-\, p)e^{uP}[u\delta^2 P + u^2(\delta P)^2]$$

and the ratio of the coefficients of these two terms is u and is *not* independent of u. Making the elimination of such terms there remains

$$A_1(-\, p)\delta e^{uP} + A_0(-\, p)e^{uP} = 0,$$

$$uA_1(-\, p)\delta P + A_0(-\, p) = 0.$$

Then $A_1(-\, p) = A(-\, p)$ will be independent of u while $A_0(-\, p) = uB(-p)$ where B is also independent of u. Since in the case considered $e^{uP(0)} = \beta_u(\infty) = 1$, $P(0)$ must $= 0$ and

$$P(p) = \int_0^{-p} \frac{B(q)}{A(q)} dq.$$

Correspondingly we have for $f_u(y)$ the equation

$$A(D)(yf) + uB(D)f = 0.$$

In general the simplest form of solution of this equation is given by definite integrals of type 4(6),* so that we do not appear to have made much progress. However if dP/dp is a rational function of p it may be resolved into a sum of terms of types $a_n p^n$, $b_n/(p - p_0)^n$ and if β_u, γ_u correspond to two functions P_1, P_2 there will be a solution $\delta_u = \beta_u \cdot \gamma_u$ corresponding to $P_1 + P_2$ so that more general solutions can be built out of a few simpler types.

Pearson† has suggested types of statistical distributions which are more general than the "normal" or Gaussian distribution. They are such as to satisfy a differential equation of the form

$$\frac{1}{f}\frac{df}{dy} = \frac{a_1 - y}{b_0 + b_1 y + b_2 y^2}.$$

When this equation is cleared of fractions there appears a term $b_2 y^2 Df$ and in such a case f cannot lead to a solution of 3(5). We must therefore choose $b_2 = 0$, and then

$$(b_1 D + 1)(yf) + (b_0 D - a_1 - b_1)f = 0.$$

Comparing with the form already given

$$A(D) = b_1 D + 1, \qquad B(D) = (b_0 D - [a_1 + b_1])/u = c_0 D - c_1.$$

$$P(p) = \int_0^{-p} \frac{c_0 q - c_1}{b_1 q + 1} dq.$$

5(1). If $b_1 = 0$, $P = \frac{1}{2} c_0 p^2 + c_1 p$, $Df/f = (c_1 u - y)/c_0 u$ and $\log f = -y^2/2c_0 u + (c_1/c_0)y + C$, where C is some constant, which must be chosen so that $\beta_u(\infty) = \int_{-\infty}^{+\infty} f_u(y)dy = 1$.

In fact this gives the normal distribution

$$f_u(y) = \frac{1}{\sqrt{(2\pi c_0 u)}} e^{-(y - c_1 u)^2/2c_0 u}$$

in which the average is at $c_1 u$ (i.e. moving with velocity c_1), and the standard deviation is $\sqrt{(c_0 u)}$.

* Cf. A. R. Forsyth, "Treatise on Differential Equations" (1903), 3d ed., p. 250.

† K. Pearson, *Philosophical Transactions of the Royal Society*, 186A, p. 343.

$$\varphi(p, \beta_u) = \int_{-\infty}^{+\infty} e^{py} f_u(y) dy$$

$$= \frac{1}{\sqrt{(2\pi c_0 u)}} e^{\frac{1}{2}c_0 u p^2 + c_1 u p} \int_{-\infty}^{+\infty} e^{-(y - c_1 u - c_0 u p)^2/2c_0 u} dy$$

$$= e^{uP}.$$

This proves that here is a possible solution.

5(2). If $b_1 \neq 0$, let $p_0 = 1/b_1$, $n = c_1 p_0 + c_0 p_0^2$, then

$$P(p) = - c_0 p_0 p - n \log (1 - p/p_0),$$

$$\log f = - p_0 y + (nu - 1) \log (y + u c_0 p_0) + C,$$

where C is some constant chosen so that $\beta_u(\infty) = 1$.

From the form of this solution it is evident that $y > - u c_0 p_0$, or in other words that $f_u(y) = 0$, when $y < - u c_0 p_0$.

When $y > - u c_0 p_0$, $f_u(y) = e^{-p_0 y}(y + u c_0 p_0)^{nu-1} p_0^{nu} e^{-u c_0 p_0^2}/\Gamma(nu)$. In this case there is a sharp boundary at $y = - u c_0 p_0$, which is therefore moving with velocity $- c_0 p_0$. The average is at $(nu/p_0) - u c_0 p_0$, the standard deviation is $\sqrt{(nu)}/p_0$.

By choosing C to be partly imaginary, $C = C' + (nu - 1) \log (- 1)$, we can obtain another solution in which $f_u(y) = 0$ when $y > - u c_0 p_0$. For the convergence of the integral p_0 must now be negative, $= - q_0$ and then when $y < u c_0 q_0$, $f_u(y) = e^{q_0 y}(u c_0 q_0 - y)^{nu-1} q_0^{nu} e^{-u c_0 q_0^2}/\Gamma(nu)$. This distribution is the mirror image of the other, reflection taking place at $y = - u c_0 p_0$ and with $q_0 = - p_0$ in place of p_0. The two cannot be combined since in one case p_0 is positive, in the other negative. When a comparison is made of the two types of solution we see that type 5(1) is continuous and as soon as u becomes positive there is a finite non-zero density of distribution in both directions; type 5(2) has a discontinuity at a moving boundary and the distribution has non-zero density on one side only of this boundary. Also in 5(2) the rate of decrease in density is less rapid than in 5(1). To make a true comparison of these rates it is necessary to make allowance for the different deviations. Let $\bar{y}$ be the average y, σ the standard deviation and put $y = \bar{y} + r\sigma$ where r is some pure number greater than 1. In case 5(1) $\bar{y} = c_1 u$, $\sigma = \sqrt{(c_0 u)}$, and

$$- \sigma Df/f = \sqrt{(c_0 u)} \left[c_1 u + r\sqrt{(c_0 u)} - c_1 u\right]/c_0 u$$

$$= r.$$

In case 5(2) $\bar{y} = nu/p_0 - u c_0 p_0$, $\sigma = \sqrt{(nu)}/p_0$, and

$$- \sigma Df/f = \frac{\sqrt{(nu)}}{p_0}\left[p_0 + \frac{1 - nu}{nu/p_0 + r\sqrt{(nu)}/p_0}\right]$$

$$= \frac{1 + r\sqrt{(nu)}}{r + \sqrt{(nu)}}$$

If $r > 1$ this is $< r$ and if r is large while u is small it is approximately $1/r$.

As the next step we might consider differential equations for f which are of the second order, that is in which dP/dp is a fraction in which either the denominator or numerator (or both) is quadratic in p. If the denominator reduces to a constant we have a case similar to 5(1) but in which P contains a term in p^3. The resulting solution for f is not elementary. If the denominator is linear, P is the sum of terms of type 5(1) and 5(2), the resulting solution is an integral composition of 5(1) and 5(2) and is not elementary.

Finally if the denominator is quadratic, P is the sum of types 5(2)and the combination in f is again not elementary.

There are an endless number of other possible solutions but the author has not had the good fortune to find types which should be so simple that they could be used as a basis for further investigation. Even the type 5(2), simple though it is, does not seem to be readily applicable to problems in dynamic probability. It may, however, be useful in exceptional cases.

For the remainder of this paper we shall use the solution 5(1) in the forms: $\beta_u(y) = \int_{-\infty}^{y} f_u(y)dy$, where

5(3) $$f_u(y) = \frac{1}{\sqrt{(2\pi Mu)}} e^{-(y-Uu)^2/2Mu},$$

5(4) $$f_u(y) = \sqrt{(R/2\pi u)}e^{-Ry^2/2u}e^{Fu}e^{-uF^2/2R},$$

$$R = 1/M, \qquad F = RU, \qquad U = MF.$$

U is the velocity of the average change in x, or we may call U the *drift* of the group.

M is the rate of increase of the average square deviation and is called the *mobility*. The standard deviation in an interval u is $\sqrt{Mu}$.

F is the ratio of drift to mobility and is called the *force*.

R is the reciprocal of the mobility and is called the *resistance*.

The formulæ 5(3), 5(4) are really identical but in some cases one or the other form is more convenient and natural.

6. *Variable Characteristics.*—We now consider the motion of a group during finite intervals of time and over finite intervals in the variable. We assume that the motion is the result of infinitesimal movements of the

type 5(3), that is to say, that when u is sufficiently small the motion is as close to that of 5(3) as we choose but that the characteristics U, M (or F, R) vary both with initial position x and initial time t. It is also assumed that, except possibly at a boundary, the first and second partial derivatives with respect to x of these characteristics exist and are limited, and that U, M, F, R are also limited.

Let y refer to a displacement from an initial position x, u a positive increase in the time t and denote $f_u(y)$ by

$$f(y,\ u;\ x,\ t) = \sqrt{(R/2\pi u)}e^{-R(y-Uu)^2/2u},$$

where R, U are possibly functions of x, t. Then

$$\int_{-\infty}^{+\infty} f(y,\ u;\ x,\ t)dy = 1,$$

$$\int_{-\infty}^{+\infty} f(y,\ u;\ x,\ t)ydy = Uu,$$

$$\int_{-\infty}^{+\infty} f(y,\ u;\ x,\ t)y^2dy = Mu,$$

$$\int_{-\infty}^{+\infty} f(y,\ u;\ x,\ t)y^{n+1}dy = 0, \qquad n > 1,$$

if we neglect terms of the second, and higher, order in u.

In an individual case the number of members of the group lying in a given interval will be an integer and the distribution will be completely discontinuous. But our problem is that of probable, or average, not actual distribution and such a probable number may be fractional or even irrational. We assume that the probable number of members of the group in the interval x to $x + dx$ is $N(x, t)dx$, that $N(x, t)$ is limited and summable from $-\infty$ to $+\infty$ and that it possesses bounded first and second partial derivatives with respect to x.

If $K(u)$ is the number crossing the point x in the direction of x increasing,

$$\begin{aligned} K(u) &= \int_{-\infty}^{x} N(s,\ t)ds \int_{x-s}^{\infty} f(y,\ u;\ s,\ t)dy \\ &= \int_{0}^{\infty} dy \int_{x-y}^{x} f(y,\ u;\ s,\ t)N(s,\ t)ds. \end{aligned}$$

This change of order of integration is legitimate since $N(s, t)$ is summable in s and

$$|f(y,\ u;\ s,\ t)| \leq \sqrt{R_1/2u}e^{-R_2y^2/2u}e^{F_1y},$$

where R_1 is the maximum, R_2 the minimum value of R, F_1 the maximum value of $F = RU$, the expression on the right-hand side being independent of s and summable in y.

If $x - p$ is substituted for s we obtain

$$K(u) = \int_0^\infty dy \int_0^y f(y,\, u;\; x-p,\, t)N(x-p,\, t)dp.$$

Expanding in powers of p,

$$\begin{aligned} f(y,\, u;\; x-p,\, t)N(x-p,\, t) &= f(y,\, u;\; x,\, t)N(x,\, t) \\ &\qquad - p\frac{\partial}{\partial x}[f(y,\, u;\; x,\, t)N(x,\, t)] + \frac{p^2}{2}\epsilon(y,\, p),\end{aligned}$$

$$\begin{aligned} |\epsilon(y,\, p)| &\leq \max_{p=0 \text{ to } y} \left| \frac{\partial^2}{\partial x^2}[f(y,\, u;\; x-p,\, t)N(x-p,\, t)] \right| \\ &\leq (a_0 + a_1 y^2/u + a_2 y^4/u^2)(1 + b_1 y + b_2 u)\sqrt{(R_1/2u)}e^{-R_2 y^2/2u}e^{F_1 y},\end{aligned}$$

where a_0, a_1, a_2, b_1, b_2 are certain constants depending on the bounds of the derivatives of U, M, F, R but are independent of y and u. From this inequality it follows that

$$\left| \int_0^\infty dy \int_0^y \frac{1}{2}p^2\epsilon(y,\, p)dp \right| \leq \text{a term of order } u^2.$$

It can be neglected and considering terms of order u, or less,

$$\begin{aligned} K(u) &= \int_0^\infty dy \int_0^y f(y,\, u;\; x,\, t)N(x,\, t)dp \\ &\qquad - \int_0^\infty dy \int_0^y \frac{\partial}{\partial x}\left[N(x,\, t) \int_0^\infty f(y,\, u;\; x,\, t)dp \right] \\ &= N(x,\, t)\int_0^\infty yf(y,\, u;\; x,\, t)dy - \frac{1}{2}\frac{\partial}{\partial x}\left[N(x,\, t)\int_0^\infty y^2 f(y,\, u;\; x,\, t)dy \right].\end{aligned}$$

Similarly the number $L(u)$ crossing in the other direction is

$$\begin{aligned} L(u) &= \int_x^\infty N(s,\, t)ds \int_{-\infty}^{x-s} f(y,\, u;\; s,\, t)dy \\ &= \int_{-\infty}^0 dy \int_x^{x-y} f(y,\, u;\; s,\, t)N(s,\, t)ds \\ &= -\, N(x,\, t)\int_{-\infty}^0 yf(y,\, u;\; x,\, t)dy + \frac{1}{2}\frac{\partial}{\partial x}\left[N(x,\, t)\int_{-\infty}^0 y^2 f(y,\, u;\; x,\, t)dy \right].\end{aligned}$$

Combining these two, the net resultant number crossing to the right (x increasing) is $K(u) - L(u)$ or

$$N(x, t) \int_{-\infty}^{+\infty} yf(y, u; x, t)dy - \frac{1}{2}\frac{\partial}{\partial x}\left[N(x, t) \int_{-\infty}^{+\infty} y^2 f(y, u; x, t)dy \right]$$
$$= NUu - \frac{1}{2}\frac{\partial}{\partial x}(NMu).$$

6(1). The net rate at which members cross at x in the direction of x increasing is given by

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM).$$

6(2). Taking into consideration the flow across a neighboring point $x + dx$, if there are no sources or sinks,

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}(NU) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(NM).$$

This equation is similar to that governing the flow of heat along a bar, but there is an added term depending on drift.

For statistically steady *flow* $\partial N/\partial t = 0$,

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) \text{ is independent of } x.$$

For statistical equilibrium (statistically stationary state)

$$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0$$

of which the general solution is $N = (C/M)e^{2\phi}$ where C is an arbitrary constant, φ is the attractive potential defined so that $\partial\varphi/\partial x = F = RU = U/M$, and U, M are now assumed to be independent of t.

It is evident that N tends to be large where the attractive potential is high or where the mobility is small.

In economics φ may be regarded as a possible substitute for "desirability" or "ophelimity."

7. *Boundary Conditions.*—At a rigid boundary there will not only be a resultant 0 flow but the flow will be 0 separately in each direction. Suppose that at x there is a distribution with finite density for smaller values of x and density 0 for larger values of x; then $L(u) = 0$ because $N(s, t) = 0$, $s > x$. Neglecting terms of order higher than u, we have seen that

$$K(u) = N \int_0^{\infty} yfdy - \frac{1}{2}\frac{\partial}{\partial x}\left[N \int_0^{\infty} y^2 fdy \right].$$

Let $y = Uu + z\sqrt{2Mu}$, then

$$\int_0^\infty yfdy = \frac{1}{\sqrt{\pi}}\int_{-U\sqrt{(u/2M)}}^{\infty}(Uu + z\sqrt{2Mu})e^{-z^2}dz$$

$$= [\sqrt{(Mu/2\pi)} + \tfrac{1}{2}Uu] + \text{terms of higher order in } u.$$

$$\int_0^\infty y^2fdy = \frac{1}{\sqrt{\pi}}\int_{-U\sqrt{(u/2M)}}^{\infty}(Uu + z\sqrt{2Mu})^2e^{-z^2}dz$$

$$= \tfrac{1}{2}Mu + \text{terms of higher order.}$$

Therefore to order u,

$$K(u) = \tfrac{1}{2}u\left[NU - \frac{1}{2}\frac{\partial}{\partial x}(NM)\right] + \sqrt{u}N\sqrt{(M/2\pi)}.$$

This shows that the flow to the right consists of half the net statistical flow which would occur if the distribution were uniform through x, instead of being bounded at x, together with a flow in which the initial quantity flowing in time u is proportional to $\sqrt{u}$. The latter constitutes a sudden gush from the region of finite density into the empty space. At an impassable boundary this gushing flow must also vanish and the necessary conditions will be

$$M = 0, \qquad NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0.$$

The condition $M = 0$ makes $R = \infty$ and this violates the conditions imposed in paragraph 6, so that we cannot conclude immediately that these are also sufficient.

If at the boundary x_0, $M = 0$ every member of the group which happens to arrive at x_0 will have immediately thereafter a velocity $U = U_0$ and the dispersion will be 0. It would appear to be a sufficient condition that U_0 should be non-positive, for then the group would not pass x_0. But since at any point $x < x_0$, there is a flow of rate $NU - \frac{1}{2}(\partial/\partial x)(NM)$ there will be a sudden change in the density between x and x_0 unless this expression also approaches 0 at $x = x_0$. Since $M_0 = 0$, this condition is equivalent to $U_0 = \frac{1}{2}(\partial M/\partial x)_0$. Now M is positive except at $x = x_0$ and if it possesses a derivative at $x = x_0$, $(\partial M/\partial x)_0$ must be non-positive, or U_0 will be non-positive. It therefore appears that the two conditions

$$M = 0, \qquad NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) \doteq 0$$

are sufficient as well as necessary. Let $W = U - \frac{1}{2}(\partial M/\partial x)$, then at $x = x_0$, $M = 0$, $W = 0$. Now

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}\left[NW - \tfrac{1}{2}M\frac{\partial N}{\partial x}\right] = -N\frac{\partial W}{\partial x} + \left(\frac{1}{2}\frac{\partial M}{\partial x} - W\right)\frac{\partial N}{\partial x} + \tfrac{1}{2}M\frac{\partial^2 N}{\partial x^2}.$$

At $x = x_0$ this becomes

$$\frac{\partial N_0}{\partial t} = -N_0(\partial W/\partial x)_0 + \tfrac{1}{2}(\partial M/\partial x)_0(\partial N/\partial x)_0.$$

It may happen that the boundary produces its effect by a "repulsion" which is considerable at some distance, but consider the case where U, M are practically unaffected until points very near x_0 are reached. Then in general M will suddenly drop to 0 from a finite value and $(\partial M/\partial x)_0$ will be large. If $(\partial N/\partial x)_0$ does not happen to be 0, $\partial N_0/\partial t$ will be large numerically unless there is some relation of the type $\partial N/\partial x = 2kN$, where k is a finite number such that $\partial W/\partial x - k\partial M/\partial x$ remains finite. Now if at the boundary *point* x_0 there were statistical equilibrium without any *real* boundary we should have $\partial N/\partial x = 2(W/M)N$. Choose $k = W'/M'$, where W', M' are the values W, M would have in the absence of a boundary. Since these would not vary very rapidly near x_0, the conditions would be satisfied. Hence we may write the general equations for our form of dynamic probability in the form:

7(1) $$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial x}(NU) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(NM)$$

throughout the range, together with the boundary conditions:

7(2) $$NU - \frac{1}{2}\frac{\partial}{\partial x}(NM) = 0,$$

in which U, M take the values which they would have in the absence of any boundary. It is important, however, to remember that in a small neighborhood of a boundary U and M actually vary rapidly and that their values will differ from those which they would possess in the absence of a boundary.

Since $W_0 = 0 = U_0 - \frac{1}{2}(\partial M/\partial x)_0$, $F_0 = R_0U_0 = \frac{1}{2}\left(\frac{1}{M}\frac{\partial M}{\partial x}\right)_0$, there will be a strong repulsive "force" very close to a boundary. The author hopes to obtain some interesting results by an extension of the analysis to two or more dimensions.

RICE INSTITUTE,
HOUSTON, TEXAS.